http://dx.doi.org/10.35596/1729-7648-2022-20-5-21-30

Original paper

UDC 681.51

DUAL CONTROL OF THE EXTREMAL MULTIDIMENSIONAL REGRESSION OBJECT

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Submitted 4 March 2022

Abstract. The statement of the problem of the dual control of the regression object with multidimensional-matrix input and output variables and dynamic programming functional equations for its solution are given. The problem of the dual control of the extremal regression object, i.e. object response function of which has an extremum, is considered. The purpose of control is reaching the extremum of the output variable by sequential control actions in production operation mode. In order to solve the problem, the regression function of the object is supposed to be quadratic in input variables, and the inner noise is supposed to be Gaussian. The sequential solution of the functional dynamic programming equations is performed. As a result, the optimal control action at the last control step is obtained. It is shoved also that the optimal control actions obtaining at the other control steps is connected with big difficulties and impossible both analytically and numerically. The control action obtained at the last control step is proposed to be used at the arbitrary control step. This control action is called the control action with passive information accumulation. The dual control algorithm with passive information accumulation was programmed for numerical calculations and tested for a number of objects. It showed acceptable results for the practice.

Keywords: dual control, multidimensional-matrix regression object, dynamic programming, passive information accumulation, extremal control systems.

Conflict of interests. The authors declare no conflict of interests.

For citation. Mukha V.S., Kako N.F. Dual Control of the Extremal Multidimensional Regression Object. Doklady BGUIR. 2022; 20(5): 21-30.

Introduction

The problem of the dual control of the multidimensional regression object is formulated as follows [1–5]. The control system with controlled object *O*, controller *C*, feedback path and driving action g_s is considered (Fig. 1).

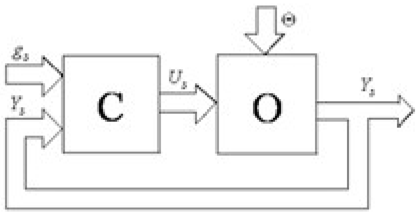

Fig. 1. To the statement of the dual control problem

The controlled object O is described at the instant of time s by the probability density function $f_{Y_s}(y_s, \Theta, U_s)$, $s = 0, 1, 2, ..., n$,

where $Y_s = (Y_{i_1,i_2,...,i_p,s})$ is the p-dimensional matrix of the output of the object at the instant of time s, $U_s = (U_{i_1,i_2,...,i_q,s})$ is the q-dimensional matrix of the input of the object at the instant of time s (control action), $\Theta = \{\Theta_1,..., \Theta_m\}$ is a set of the parameters of the controlled object consisting of the random multidimensional matrices $\Theta_1,..., \Theta_m$ with known priory joint probability density function $f_{\Theta,0}(\theta)$. We will call the set $\Theta = \{\Theta_1,..., \Theta_m\}$ a generalized parameter of the object O. It is supposed, that the generalized parameter Θ takes constant value for all of the instants of time $s = 0,1,...,n$. The driving action g_s is supposed to be known deterministic multidimensional-matrix sequence.

The quality of the functioning of the system at each instant of time s is estimated by a specific loss function $W_s(Y_s, g_s)$, depending of output Y_s and, might, driving action g_s. A system, for which the total for $n+1$ instants of time total average risk

$$R = E\{\sum_{s=0}^{n} W_s(Y_s, g_s)\} = \sum_{s=0}^{n} R_s \text{ , } R_s = E(W_s(Y_s, g_s)), \tag{1}$$

is minimal, is called optimal system. There $E(\cdot)$ means the mathematical expectation, $R_s = E(W_s(Y_s, g_s))$ is a specific risk. The control action U_s belongs to some permissible area. The controller C uses all of the past information in the form of observations $\vec{u}_{s-1} = (u_0, u_1,..., u_{s-1})$, $\vec{y}_{s-1} = (y_0, y_1,..., y_{s-1})$ of the input and output values of the object to determine the control action u_s at the instant of time s.

The task consists of determining the strategies of the controller C, i. e. sequence of the conditional probability density functions $f_{U_s}(u_s / \vec{u}_{s-1}, \vec{y}_{s-1})$, $i = 0,1,...,n$, for which the total average risk R (1) is minimal.

As it is known [2–5], the optimal strategies of the controller C are not randomized, i.e. the control actions U_s are not random and will be denoted u_s. In this conditions the controller C will be described by conditional probability density function $f_{Y_s}(y_s / \theta, u_s)$, where u_s is the fixed value of the variable U_s. We will use the following simplified notation: $f_{\Theta,0}(\theta) = f_0(\theta)$, $f_{Y_s}(y_s / \theta, u_s) = f(y_s / \theta, u_s)$.

The optimal control algorithm, i.e. the sequence of the control actions $u_n, u_{n-1},..., u_0$ is determined in pointed inverse order from the following functional equations:

$$f_n^*(\vec{u}_{n-1}, u_n^*, \vec{y}_{n-1}) = \min_{u_n \in \bigcup} \varphi_n(\vec{u}_n, \vec{y}_{n-1}), \tag{2}$$

$$f_{n-m}^*(\vec{u}_{n-m-1}, u_{n-m}^*, \vec{y}_{n-m-1}) = \min_{u_{n-m} \in \bigcup} [\varphi_{n-m}(\vec{u}_{n-m}, \vec{y}_{n-m-1}) +$$
$$+ \int_{\Omega(y_{n-m})} f_{n-m+1}^*(\vec{u}_{n-m}, u_{n-m+1}^*, \vec{y}_{n-m}) f(y_{n-m} / \vec{u}_{n-m}, \vec{y}_{n-m-1}) d\Omega], \; m = 1,2,...,n, \tag{3}$$

where φ_s is determined by expression

$$\varphi_s(\vec{u}_s, \vec{y}_{s-1}) = \int_{\Omega(\vec{y}_s)} W_s(y_s, g_s) f(y_s / \vec{u}_s, \vec{y}_{s-1}) d\Omega, \; s = 0,...,n, \tag{4}$$

in which $f(y_s / \vec{u}_s, \vec{y}_{s-1}) = \int_{\Omega(\theta)} f(y_s / \theta, u_s) f_s(\theta) d\Omega$, (5)

$$f_s(\theta)=\frac{f_0(\theta)\prod_{v=0}^{s-1}f(y_v/\theta,u_v)}{\int\limits_{\Omega(\theta)}f_0(\theta)\prod_{v=0}^{s-1}f(y_v/\theta,u_v)d\Omega}, \quad (6)$$

and u^*_{n-m+1} is optimal control action for the instant of time $(n-m+1)$.

Note. The notation $\min\limits_{u_n\in\mho}\varphi_n(\vec{u}_n,\vec{y}_{n-1})$ means the following:

$$\min_{u_n\in\mho}\varphi_n(\vec{u}_n,\vec{y}_{n-1})=\varphi_n(u^*_n,\vec{u}_{n-1},\vec{y}_{n-1}).$$

Dual control of the extremal regression object

Let us consider the case of dual control when the controlled object has an extremal characteristic, and the task consist of the search and support this extremal state. The task is concretized in this case as follows.

The controlled object is described at the instant of time s by the gaussian probability density function:

$$f(y_s/c,u_s)=N(\psi(c,u_s),d_Y), \quad (7)$$

where $\psi(c,u_s)$ is the regression function, $d_Y>0$ is the variance of the inner noise, u_s is the q-dimensional matrix of the control action $u_s=(u_{j_{(q)},s})$, $j_{(q)}=(j_1,j_2,...,j_q)$, y_s is the scalar variable (p-dimensional matrix with $p=0$), c is a some set of the parameters (generalized parameter of the object). Let us note that we natation now the generalized paramener as c instead of θ in expressions (5), (6). We suppose too that the regression function is quadratic:

$$y=\psi(C,u)=\sum_{i=0}^{m}{}^{0,iq}(C_iu^i)=\sum_{i=0}^{m}{}^{0,iq}(u^iC_{t,i})=\psi(C_t),\ m=2, \quad (8)$$

where C_i, $i=0,1,2$, are the $(p+iq)$-dimensional random matrices, at that C_2 is symmetrical relative its q-multi-indexes, $C_i=(C_{t,i})^{H_{p+iq,iq}}$, $C_{t,i}=(C_2)^{B_{p+iq,iq}}$, $H_{p+iq,iq}$ and $B_{p+iq,iq}$ are the substitutions of transpose of the type "back" and "onward" respectively [6]. Provided the regrassion function (8), the probability density function of the object (7) take the following form:

$$f(y_n/c_t,u_n)=\frac{1}{\sqrt{(2\pi)^{k_Y}\,|\,d_Y\,|}}\exp\left(-\frac{1}{2}\,{}^{0,p}\left(d_Y^{-1}(y_n-\sum_{i=0}^{m}{}^{0,iq}(u_n^ic_{t,i}))^2\right)\right). \quad (9)$$

For the task of the dual search of the minimum of the regression function we should to choose the loss function in the form $W(Y_s)=Y_s$.

Let us turn to the functional equations of the dual control (2)–(6) (with replasing θ by c_t) and find the control actions $u_n,u_{n-1},...,u_0$ based on these equations.

1. For first, let us find the posterior probability density function $f_n(c_t)$ of the random cell $C_t=\{C_{t,1},C_{t,2},C_{t,3}\}$ by the Bayes formula (6). We will consider the right hand part of the equality (8) with the parameters $C_{t,1},C_{t,2},C_{t,3}$ and will suppose the general case, when the output variable y is p-dimensional matrix. Then $C_{t,i}$ are $(iq+p)$-dimensional matrices, C_i are $(p+iq)$-dimensional matrices, at that $C_{t,i}=(C_i)^{T_i}$, $T_i=B_{p+iq,iq}$, and $C_i=(C_{t,i})^{T'_i}$, $T'_i=H_{p+iq,iq}$, $i=0,1,2$.

Let us agree to use below the following notations: $i_1, i_2, \ldots$ are separate indexes, $\bar{i}_{(p)} = (i_1, i_2, \ldots, i_p)$ is the set of p indexes (p-multi-index); $\bar{\bar{i}}_{(p,k)} = (\bar{i}_{(p),1}, \bar{i}_{(p),2}, \ldots, \bar{i}_{(p),k})$ is the set of k p-multi-indexes.

Let the random cell $C_t = \{C_{t,k}\}$, $k = \overline{0,m}$, has the Gaussian priory probability denity function described by the following expresion [7]:

$$f(c_t) = M_{\Xi} \exp(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}({}^{0,q_i}((c_{t,i} - \nu_{c_t,i})d_{c_t}^{i,j})(\xi_j - \nu_{c_t,j})) =$$

$$= M_{c_t} \exp\left\{-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}({}^{0,q_i}(c_{t,i}d_{c_t}^{i,j})c_{t,j}) + \sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}({}^{0,q_i}(c_{t,i}d_{c_t}^{i,j})\nu_{c_t,j}) - \frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}({}^{0,q_i}(\nu_{c_t,i}d_{c_t}^{i,j})\nu_{c_t,j})\right\},$$

$$M_{c_t} = \frac{1}{\sqrt{(2\pi)^{n_c} \, |d_{c_t}|}}, \; q_i = p + iq, \; i = \overline{0,m},$$

where the two-dimensional cell $d_{c_t} = \{d_{c_t,i,j}\}$, $i, j = \overline{0,m}$, is the variance-covriance cell of the random cell C_t [7], $d_{c_t,i,j} = E\left({}^{0,0}\left((C_{t,i} - \nu_{c_t,i})(C_{t,j} - \nu_{c_t,j})\right)\right)$ is the $((iq+p)+(jq+p))$-dimensional matrix, $d_{c_t}^{-1} = \{d_{c_t}^{i,j}\}$, $i, j = \overline{0,m}$, is the cell invere to the cell d_{c_t}, $\nu_{c_t} = \{\nu_{c_t,0}, \nu_{c_t,1}, \ldots, \nu_{c_t,m}\} = \{\nu_{c_t,i}\}$, $i = \overline{0,m}$, is the one-dimensional cell of the mathematical expectation of the random cell C_t, i.e. $\nu_{c_t,i} = E(C_{t,i})$ is the $(iq+p)$-dimensional matrix, n_c is the number of the scalar elements of the cell C_t. Then the posterior probability density function $f_n(c)$ (6) is defined by the following expression [7]:

$$f(c_t / \vec{y}_{n-1}, \vec{u}_{n-1}) = \frac{1}{\sqrt{(2\pi)^{n_y} \, |D_{c_t}|}} \exp\left(-\frac{1}{2}{}^{0,2}\{D_{c_t}^{-1\,0,0}\{c_t - \mathrm{N}_{c_t}\}^2\}\right) = f_n(c_t), \tag{10}$$

in which $D_{c_t} = \{D_{c_t,i,j}\}$,

$$D_{c_t}^{-1} = \{D_{c_t}^{i,j}\} = \{d_{c_t}^{i,j} + S_{i,j}\} = \{d_{c_t}^{i,j} + ({}^{0,0}(d_Y^{-1} S_{u^i u^j}))^{T_{i,j}}\}, \; i, j = \overline{0,m}, \tag{11}$$

$$B = \{B_i\} = \left\{\sum_{j=0}^{m} {}^{0,jq+p}(d_{c_t}^{i,j}\nu_{c_t,j}) + {}^{0,p}(d_{\varepsilon}^{-1} S_{yu^i})^{T_i}\right\}, \; i = \overline{0,m}, \tag{12}$$

$$N_{c_t} = \{N_{c_t,i}\} = {}^{0,1}\{D_{c_t} B\} = \{\sum_{j=0}^{m} {}^{0,p+jq}(D_{c_t,i,j} B_j)\}, \; i = \overline{0,m}, \tag{13}$$

$$S_{u^k u^\lambda} = \sum_{\mu=1}^{n-1} {}^{0,0}(u_\mu^k u_\mu^\lambda), \; S_{yu^\lambda} = \sum_{\mu=1}^{n-1} {}^{0,0}(y_\mu u_\mu^\lambda), \tag{14}$$

$\vec{y}_{n-1} = (y_1, y_2, \ldots, y_{n-1})$, $\vec{u}_{n-1} = (u_1, u_2, \ldots, u_{n-1})$.

The substitutions of transpose $T_{i,j}$ in (11) and T_i in (12) have the following forms:

$$T_{i,j} = \begin{pmatrix} \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_i, \bar{\lambda}, \bar{j}_1, \bar{j}_2, \ldots, \bar{j}_j, \bar{\mu}, \\ \bar{\lambda}, \bar{\mu}, \bar{i}_1, \bar{i}_2, \ldots, \bar{i}_i, \bar{j}_1, \bar{j}_2, \ldots, \bar{j}_j \end{pmatrix}, \; i, j = \overline{0,m},$$

$T_i = \begin{pmatrix} \bar{i}_1, \bar{i}_2, ..., \bar{i}_i, \overline{\mu} \\ \overline{\mu}, \bar{i}_1, \bar{i}_2, ..., \bar{i}_i \end{pmatrix}$, $i = \overline{0, m}$, where the multi-indexes $\bar{j}_1, \bar{j}_2, ..., \bar{j}_j, \bar{i}_1, \bar{i}_2, ..., \bar{i}_i$ contain by q indexes and the multi-indexes $\overline{\lambda}, \overline{\mu}$ contain by p indexes. There are no multi-indexes $\overline{\lambda}, \overline{\mu}$ in these substitutions in the case of $p = 0$, and substitutions $T_{i,j}$, T_i in this case are identical [6].

The two-dimensional cell $D_{c_t}^{-1} = \{D_{c_t}^{i,j}\}$, $i, j = \overline{0, m}$, (11) has the same dimensions as the two-dimensional cell $D_{c_t} = \{D_{c_t,i,j}\}$, i.e. $D_{c_t}^{i,j}$ is the $((iq+p)+(jq+p))$-dimension matrix.

The element B_i of the one-dimension cell $B = \{B_i\}$, $i = \overline{0, m}$, (12) is the $(iq + p)$-dimension matrix. It is of interest in dual control to use the single measurements for updating the estimations (10)–(14).

We will have for this the following expressions:

$S_{u_s^k u_s^\lambda} = {}^{0,0}(u_s^{k+\lambda})$, $S_{y_s u_s^\lambda} = {}^{0,0}(y_s u_s^\lambda)$,

determined by single measurement (u_s, y_s), instead of the expressions (14).

2. Secondly, let us find the probability density function $f(y_n / \vec{u}_n, \vec{y}_{n-1})$ by the formula (5),

$$f(y_n / \vec{u}_n, \vec{y}_{n-1}) = \int_{\Omega(C)} f(y_n / c_t, u_n) f_n(c_t) d\Omega, \tag{15}$$

where $f_n(c_t)$ is determined by the formula (10). We will use for this the following theorem from [7]:

Theorem (total probability formula for the joint Gaussian distribution of the multidimensional random matrices). Let $\Xi = \{\Xi_i\}$, $i = 1, 2, ..., m$, be an one-dimensional random cell, composed of the q_i-dimensional matrices Ξ_i, k_i the number of the scalar components of the matrix Ξ_i, $f(\xi)$ the probability density function of the cell Ξ, $k_\Xi = k_1 + k_2 + ... + k_m$ the number of the scalar components of the cell Ξ, $f(y / \xi)$ the condition probability density function of a p-dimensional matrix Y, k_Y the number of the scalar components of the matrix Y, E^{k_Ξ} the k_Ξ-dimensional Euclidean space. If in the total probability formula

$$f(y) = \int_{E^{k_\Xi}} f(y / \xi) f(\xi) d\xi \tag{16}$$

the conditional probability density function $f(y / \xi)$ has the following form

$$f(y / \xi) = \frac{1}{\sqrt{(2\pi)^{k_Y} |d_Y|}} \exp\left(-\frac{1}{2} {}^{0,p}(d_Y^{-1}(y - \sum_{i=1}^{m} {}^{0,q_i}(h_i \xi_i))^2) \right), \tag{17}$$

where h_i is a $(p + q_i)$-dimensional matrix, allowing the multiplication ${}^{0,q_i}(h_i \xi_i)$, and the probability density function $f(\xi)$ has the following form

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{k_\Xi} |d_\Xi|}} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_j}({}^{0,q_i}((\xi_i - \nu_{\Xi_i}) d_\Xi^{i,j})(\xi_j - \nu_{\Xi_j})) \right\},$$

then the integral (6) (the total probability formula) is defined by the following expression:

$$f(y) = \int_{E^{k_\Xi}} f(y / \xi) f(\xi) d\xi = \frac{1}{\sqrt{(2\pi)^{k_Y} |D_Y|}} \exp\left(-\frac{1}{2} {}^{0,p}(D_Y^{-1}(y - \sum_{i=1}^{m} {}^{0,q_i}(h_i \nu_{\Xi,i}))^2) \right), \tag{18}$$

where $D_Y = d_Y + \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_j}({}^{0,q_i}(h_i d_{\Xi,i,j}) h_j)$.

Let us replace ξ by c_t and $f(\xi)$ by $f_n(c_t)$ (10) in this theorem and compare the expression (9) with the expression (17) from theorem. We realize that $p_i = iq$, $h_i = u_n^i$. In accordance with formula (18) of the theorem we obtain the following expression for the integral (15):

$$f(y_n / \vec{u}_n, \vec{y}_{n-1}) = \int\limits_{E^{k_C}} f(y_n / c_t, u_n) f_n(c_t) dc_t = \frac{1}{\sqrt{(2\pi)^{k_Y} |D_Y|}} \exp\left(-\frac{1}{2} {}^{0,p}(D_Y^{-1}(y_n - N_Y)^2) \right), \quad (19)$$

where $D_Y = d_Y + \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,jq}({}^{0,iq}(u_n^i D_{c_t,i,j}) u_n^j)$, (20)

$$N_Y = \sum_{i=0}^{m} {}^{0,iq}(u_n^i N_{c_t,i}). \quad (21)$$

The matrices $D_{c_t,i,j}$ and $N_{c_t,i}$ in (20), (21) are defined by the expressions (11), (13).

3. Thirdly, the further calculations are connected with formula (4) of the functional equations. When the loss function is $W_n = Y_n$, then we need to calculate the integral

$$\varphi_n(\vec{u}_n, \vec{y}_{n-1}) = \int\limits_{E^{n_y}} y_n f(y_n / \vec{u}_n, \vec{y}_{n-1}) dy_n,$$

with weight function $f(y_n / \vec{u}_n, \vec{y}_{n-1})$ (19).

This integral is nothing more than posterior mean value (21):

$$\varphi_n(\vec{u}_n, \vec{y}_{n-1}) = \int\limits_{R^1} y_n f(y_n / \vec{u}_n, \vec{y}_{n-1}) dy_n = \sum_{i=0}^{m} {}^{0,iq}(u_n^i N_{c_t,i}) = N_{c_t,0} + {}^{0,q}(u_n N_{c_t,1}) + {}^{0,2q}(u_n^2 N_{c_t,2}).$$

Returning to the case $p = 0$ gives the equalities $N_{c_t,1} = N_{c,1}$, $N_{c_t,2} = N_{c,2}$, $S_{u^k u^\lambda} = {}^{0,0}(u_{n-1}^k u_{n-1}^\lambda)$, $S_{yu^\lambda} = {}^{0,0}(y_{n-1} u_{n-1}^\lambda)$, identical substitutions $T_{i,j}$, T_i and the following expression for the function $\varphi_n(\vec{u}_n, \vec{y}_{n-1})$:

$$\phi_n(\vec{u}_n, \vec{y}_{n-1}) = \int\limits_{R^1} y_n f(y_n / \vec{u}_n, \vec{y}_{n-1}) dy_n = \sum_{i=0}^{m} {}^{0,iq}(u_n^i N_{c_t,i}) = N_{c_t,0} + {}^{0,q}(N_{c_t,1} u_n) + {}^{0,2q}(N_{c_t,2} u_n^2),$$

where $N_{c_t} = \{N_{c_t,i}\} = {}^{0,1}\{D_{c_t} B\} = \{\sum_{j=0}^{m} {}^{0,p+jq}(D_{c_t,i,j} B_j)\}$, $i = \overline{0,m}$.

$$D_{c_t}^{-1} = \{D_{c_t}^{i,j}\} = \{d_{c_t}^{i,j} + S_{i,j}\} = \{d_{c_t}^{i,j} + ({}^{0,0}(d_Y^{-1} u_{n-1}^{i+j}))\},\ i, j = \overline{0,m},$$

$$B = \{B_i\} = \left\{ \sum_{j=0}^{m} {}^{0,jq+p}(d_{c_t}^{i,j} \nu_{c_t,j}) + {}^{0,p}(d_Y^{-1}\ {}^{0,0}(y_{n-1} u_{n-1}^i)) \right\},\ i = \overline{0,m},$$

This function $\varphi_n(\vec{u}_n, \vec{y}_{n-1})$ has an extremum at the point [Appendix]

$$u_n^* = \arg\min_{u_n} \varphi_n(\vec{u}_n, \vec{y}_{n-1}) = -\frac{1}{2} {}^{0,q}\left({}^{0,q} N_{c_t,2}^{-1} N_{c_t,1} \right), \quad (22)$$

which is the optimal control action at the last n-th instant of time. The minimal value of the function $\varphi_n(\vec{u}_n, \vec{y}_{n-1})$ is defined by the following expression (Appendix):

$$\varphi_n(u_n^*, \vec{u}_{n-1}, \vec{y}_{n-1}) = \min_{u_n \in \mho} \varphi_n(\vec{u}_n, \vec{y}_{n-1}) = N_{c_t,0} - \frac{1}{4} {}^{0,q}\left(N_{c_t,1}\ {}^{0,q}\left({}^{0,q} N_{c_t,2}^{-1} N_{c_t,1} \right) \right). \quad (23)$$

The search of the optimal control action u_n^* at the last n-th instant of time finished there and the search of the optimal control action u_{n-1}^* at the penultimate $(n-1)$-th instant of time starts. The control action u_{n-1}^* is defined by the following expression (formula (3)):

$$u_{n-1}^* = \arg\min_{u_{n-1}\in U}\left[\varphi_{n-1}(\vec{u}_{n-1},\vec{y}_{n-2}) + \int\limits_{\Omega(y_{n-1})}\varphi_n(u_n^*,\vec{u}_{n-1},\vec{y}_{n-1})f(y_{n-1}/\vec{u}_{n-1},\vec{y}_{n-2})d\Omega\right]. \quad (24)$$

The function $\varphi_n(u_n^*,\vec{u}_{n-1},\vec{y}_{n-1})$ (23) in (24) is subjected to integration by y_{n-1} with weight function $f(y_{n-1}/\vec{u}_{n-1},\vec{y}_{n-2})$ and then is minimized by u_{n-1} in sum with the $\varphi_{n-1}(\vec{u}_{n-1},\vec{y}_{n-21})$. One can show that the calculations by the expression (24) are very difficult. Indeed, the expressions for the parameters $N_{c_t,0}$, $N_{c_t,1}$, $N_{c_t,2}$ of the function $\varphi_n(u_n^*,\vec{u}_{n-1},\vec{y}_{n-1})$ (23) have the following expanded form:

$$N_{c_t} = \left\{\begin{matrix} {}^{0,0}(D_{c_t,0,0}B_0) + {}^{0,q}(D_{c_t,0,1}B_1) + {}^{0,2q}(D_{c_t,0,2}B_2) \\ {}^{0,0}(D_{c_t,1,0}B_0) + {}^{0,q}(D_{c_t,1,1}B_1) + {}^{0,2q}(D_{c_t,1,2}B_2) \\ {}^{0,0}(D_{c_t,2,0}B_0) + {}^{0,q}(D_{c_t,2,1}B_1) + {}^{0,2q}(D_{c_t,2,2}B_2) \end{matrix}\right\} = \left\{\begin{matrix} N_{c_t,0} \\ N_{c_t,1} \\ N_{c_t,2} \end{matrix}\right\}, \quad (25)$$

where
$$D_{c_t}^{-1} = \left\{\begin{matrix} d_{c_t}^{0,0}+1 & d_{c_t}^{0,1}+u_{n-1} & d_{c_t}^{0,2}+u_{n-1}^2 \\ d_{c_t}^{1,0}+u_{n-1} & d_{c_t}^{1,1}+u_{n-1}^2 & d_{c_t}^{1,2}+u_{n-1}^3 \\ d_{c_t}^{2,0}+u_{n-1}^2 & d_{c_t}^{2,1}+u_{n-1}^3 & d_{c_t}^{2,2}+u_{n-1}^4 \end{matrix}\right\}, \quad (26)$$

$$B = \{B_i\} = \left\{\begin{matrix} {}^{0,0}(d_{c_t}^{0,0}\nu_{c_t,0}) + {}^{0,q}(d_{c_t}^{0,1}\nu_{c_t,1}) + {}^{0,2q}(d_{c_t}^{0,2}\nu_{c_t,2}) + {}^{0,0}(d_Y^{-1}y_{n-1}) \\ {}^{0,0}(d_{c_t}^{1,0}\nu_{c_t,0}) + {}^{0,q}(d_{c_t}^{1,1}\nu_{c_t,1}) + {}^{0,2q}(d_{c_t}^{1,2}\nu_{c_t,2}) + {}^{0,0}(d_Y^{-1}y_{n-1}u_{n-1}) \\ {}^{0,0}(d_{c_t}^{2,0}\nu_{c_t,0}) + {}^{0,q}(d_{c_t}^{2,1}\nu_{c_t,1}) + {}^{0,2q}(d_{c_t}^{2,2}\nu_{c_t,2}) + {}^{0,0}(d_Y^{-1}y_{n-1}u_{n-1}^2) \end{matrix}\right\} = \left\{\begin{matrix} B_0 \\ B_1 \\ B_2 \end{matrix}\right\},$$

and the matrices $D_{c_t,i,j}$, $i,j=\overline{0,m}$, in (25) are the elements of the cell D_{c_t} inverse to the cell $D_{c_t}^{-1}$ (26). One can understand, that the matrices $N_{c_t,0}$, $N_{c_t,1}$, $N_{c_t,2}$ (25) are very complicate functions of the matrix u_{n-1}. As a result, it is impossible to perform the analytical calculations and minimization in the expression (24). The numerical minimization in the expression (24) is impossible too.

However, the control action u_n^* (22), obtained at the last instant of time, can be used at any instant of time s. We will call the expression (22) the algorithm of the dual extremal control with passive information storage. Let us consider more general case $p \neq 0$ and loss function $W(Y_s) = {}^{0,p}(\alpha(Y_s - g))$, where α and g are constant p-dimensional matrices with the same dimension as the matrix Y. We have in this case the task of the dual search of the extremum of the weighted output vatiable of the regression function.

The calculation of the functions $f_n(c_t)$ and $f(y_n/\vec{u}_n,\vec{y}_{n-1})$ is described above (the functions (10), (19)).

The function $\varphi_n(\vec{u}_n,\vec{y}_{n-1})$, in accordance with the formula (4), is defined by the following expression:

$$\varphi_n(\vec{u}_n,\vec{y}_{n-1}) = \int\limits_{\Omega(\vec{y}_n)} W_n(Y_n) f(y_n/\vec{u}_n,\vec{y}_{n-1})d\Omega =$$
$$= E(W(Y_n)) = E({}^{0,p}(\alpha(Y_n - g))) = {}^{0,p}(\alpha(E(Y_n) - g)) = {}^{0,p}(\alpha(N_Y - g)).$$

Taking into account the expression (21) for the N_Y, we get:

$$\varphi_n(\vec{u}_n,\vec{y}_{n-1}) = \int\limits_{E^{n_y}} {}^{0,p}(\alpha(y_n-g))f(y_n/\vec{u}_n,\vec{y}_{n-1})dy_n = {}^{0,p}\left(\alpha\left(\sum_{i=0}^{m}{}^{0,iq}(u_n^i N_{c_t,i})-g\right)\right)=$$

$$= {}^{0,p}\left(\alpha\left(N_{c_t,0} + {}^{0,q}(u_n N_{c_t,1}) + {}^{0,2q}(u_n^2 N_{c_t,2}) - g\right)\right), \text{ and also}$$

$$\varphi_n(\vec{u}_n,\vec{y}_{n-1}) = \int\limits_{E^{n_y}} {}^{0,p}(\alpha(y_n-g))f(y_n/\vec{u}_n,\vec{y}_{n-1})dy_n = {}^{0,p}\left(\alpha\left(N_{c,0} + {}^{0,q}(N_{c,1}u_n) + {}^{0,2q}(N_{c,2}u_n^2) - g\right)\right)=$$

$$= {}^{0,p}(\alpha(N_{c,0}-g)) + {}^{0,q}({}^{0,p}(\alpha N_{c,1})u_n) + {}^{0,2q}({}^{0,p}(\alpha N_{c,2})u_n^2),$$

where $N_{c,0} = N_{c_t,0}$, $N_{c,1} = (N_{c_t,1})^{H_{p+q,q}}$, $N_{c,2} = (N_{c_t,2})^{H_{p+2q,2q}}$. Hence (Appendix)

$$u_n^* = \arg\min_{u_n} \varphi_n(\vec{u}_n,\vec{y}_{n-1}) = -\frac{1}{2}\,{}^{0,q}\left({}^{0,q}M_{c,2}^{-1}M_{c,1}\right), \tag{27}$$

where $M_{c,1} = {}^{0,p}(\alpha N_{c,1})$, $M_{c,2} = {}^{0,p}(\alpha N_{c,2})$. The formula (27) is the optimal control action at the last n-th instant of time.

The minimal value of the function $\varphi_n(\vec{u}_n,\vec{y}_{n-1})$ is (Appendix):

$$\varphi_n(u_n^*,\vec{u}_{n-1},\vec{y}_{n-1}) = \min_{u_n\in U}\varphi_n(\vec{u}_n,\vec{y}_{n-1}) = M_{c,0} - \frac{1}{4}\,{}^{0,q}\left(M_{c,1}\,{}^{0,q}\left({}^{0,q}M_{c,2}^{-1}M_{c,1}\right)\right),$$

where $M_{c,0} = {}^{0,p}(\alpha(N_{c,0}-g))$.

The search of the optimal control action u_{n-1}^* at the penultimate $(n-1)$-th instant of time seems unfeasible. The control action u_n^* (27), obtained at the last instant of time, can be used at any instant of time s.

Computer simulation

The algorithms of the optimal dual control with passive information storage (22), (27) were realized programmatically, utilized at a number of objects and showed results acceptable for practice. For instant, the object with Booth function as the regression function was simulated:

$$y = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2. \tag{28}$$

This function has minimum at the point $(x_1, x_2) = (1,3)$. The following priory characteristics of the coefficients of the approximating polynomial (8) was used: the priory mathematical expectations

$$\nu_{c_0} = 50,\ \nu_{c_1} = \begin{pmatrix}-20 & -25\end{pmatrix},\ \nu_{c_2} = \begin{pmatrix}9 & 5\\ 5 & 9\end{pmatrix}$$

and the priory variance-covariance matrices

$$d_{c,0,0} = 1,\ d_{c,1,1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ d_{c,2,2} = 0.5\cdot\begin{pmatrix} \overset{1,1,1,1}{\overbrace{1}} & \overset{1,1,2,1}{\overbrace{0}} & \overset{1,1,1,2}{\overbrace{0}} & \overset{1,1,2,2}{\overbrace{0}} \\ \overset{2,1,1,1}{\overbrace{0}} & \overset{2,1,2,1}{\overbrace{1}} & \overset{2,1,1,2}{\overbrace{0}} & \overset{2,1,2,2}{\overbrace{0}} \\ \overset{1,2,1,1}{\overbrace{0}} & \overset{1,2,2,1}{\overbrace{0}} & \overset{1,2,1,2}{\overbrace{1}} & \overset{1,2,2,2}{\overbrace{0}} \\ \overset{2,2,1,1}{\overbrace{0}} & \overset{2,2,2,1}{\overbrace{0}} & \overset{2,2,1,2}{\overbrace{0}} & \overset{2,2,2,2}{\overbrace{1}} \end{pmatrix}.$$

The four-dimensional matrix $d_{c,2,2}$ is presented by an associated with it two-dimensional matrix. The covariance matrices $d_{c,0,1}$, $d_{c,1,0}$, $d_{c,0,2}$, $d_{c,2,0}$, $d_{c,1,2}$ and $d_{c,2,1}$ are taken as zero matrices of the appropriate sizes.

The sequence of the control actions is showed in a figure for some variant of the simulation.

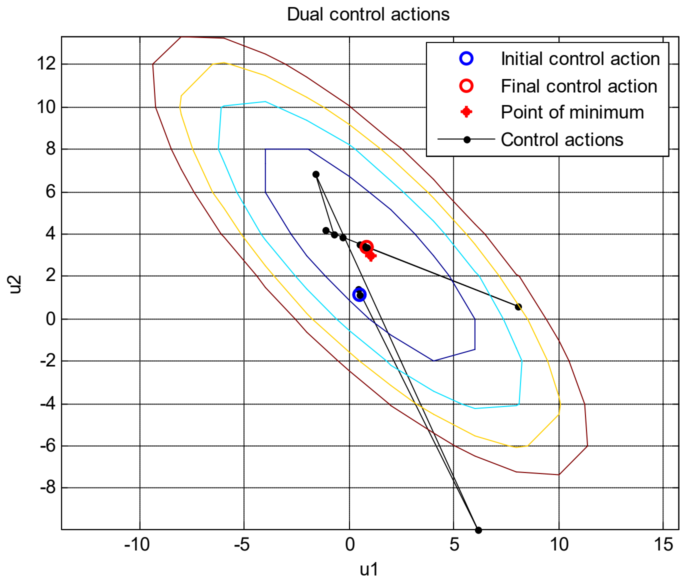

Fig. 1. The dual control actions for the example

One can see in the figure that the point of the extremum of the regression function (28) is reached.

Conclusion

To sum up, the general solution to the problem of the dual control with passive information storage of the extremal multidimensional regression object in the Gaussian case was obtained for the first time. This solution can be applied to control various technological processes, but each of them requires separate consideration. One of them is the allowance distribution problem [8].

Appendix

Let $x = (x_{j_{(q)}})$, $j_{(q)} = (j_1, j_2, ..., j_q)$, be a q-dimensional matrix, that is the argument of a p-dimensional-matrix function $y = (y_{i_{(p)}})$, $i_{(p)} = (i_1, i_2, ..., i_p)$, and this function has the form

$y = \varphi(x) = c_0 + {}^{0,q}(c_1 x) + {}^{0,2q}(c_2 x^2) = c_{t,0} + {}^{0,q}(x c_{t,1}) + {}^{0,2q}(x^2 c_{t,2})$, where c_k, $k = 0,1,2$, are the $(p + kq)$-dimensional-matrix coefficients of the function $\varphi(x)$, and c_2 is symmetric relative its last q-multi-indexes. Let it be required to find the extremum of this function.

Optimal value of x can be found from the equation $\partial\varphi(x)/\partial x = 0$. Differentiating of $\varphi(x)$ gives the equation $\hat{c}_1 + 2\,{}^{0,q}(c_2 x) = 0$.

Hence $x_n^* = -\frac{1}{2}\,{}^{0,q}({}^{0,q}c_2^{-1} c_1)$, where ${}^{0,q}c_2^{-1}$ is the matrix $(0,q)$-inverse to the matrix c_2.

Let us to find the minimum value $y^* = \varphi(x^*)$ of the function $\varphi(x)$.

Since ${}^{0,2q}(c_2 x^2) = {}^{0,q}({}^{0,q}(c_2 x)x)$ and the equation ${}^{0,q}(c_2 x_n^*) = -\frac{1}{2} c_1$ for $x = x^*$ is fullfilled, we have

$^{0,2q}(c_2(x_n^*)^2) = -\frac{1}{2}{}^{0,q}(c_1 x_n^*)$ and

$$y^* = \varphi(x^*) = c_0 + {}^{0,q}(c_1 x^*) + {}^{0,2q}(c_2(x^*)^2) = c_0 + {}^{0,q}(c_1 x^*) - \frac{1}{2}{}^{0,q}(c_1 x^*) = c_0 + {}^{0,q}(c_1 x^*).$$

Substituting x_n^* into this expression dives $y^* = c_0 - \frac{1}{4}{}^{0,q}(c_1\, {}^{0,q}({}^{0,q}c_2^{-1} c_1))$.

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Mukha V.S. developed and wrote the article.
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