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Original paper

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GEOMETRIZATION OF THE THEORY OF ELECTROMAGNETIC AND SPINOR FIELDS ON THE BACKGROUND OF THE SCHWARZSCHILD SPACETIME

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Abstract. The geometrical Kosambi–Cartan–Chern approach has been applied to study the systems of differential equations which arise in quantum-mechanical problems of a particle on the background of non-Euclidean geometry. We calculate the geometrical invariants for the radial system of differential equations arising for electromagnetic and spinor fields on the background of the Schwarzschild spacetime. Because the second invariant is associated with the Jacobi field for geodesics deviation, we analyze its behavior in the vicinity of physically meaningful singular points $r = M$, ∞. We demonstrate that near the Schwarzschild horizon $r = M$ the Jacobi instability exists and geodesics diverge for both considered problems.

Keywords: electromagnetic field, spinor field, Schwarzschild spacetime, Kosambi–Cartan–Chern invariants, Jacobi stability.

Conflict of interests. The authors declares no conflict of interests.

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Introduction

The behavior of material fields in the vicinity of cosmological objects such as black holes or neutron stars is of great interest [1, 2]. Relevant spacetime models describe gravitational potentials of these objects. However, the search for analytical solutions under the background of curved spacetimes remains to be a complicated problem that stipulates the development of other methods to analyze the behavior of the corresponding dynamical systems.

The Kosambi–Cartan–Chern geometrical approach (KCC-theory) was developed in details in numerous mathematical books and papers [3–5]. KCC-theory allowed to describe the evolution of a dynamical system in a configuration space of the Lagrange type. At that, the dynamical system is governed by the system of second-order differentials equations.

Results and discussion

$$\frac{dy^i}{dr}+2G^i=0, \qquad y^i=\frac{dx^i}{dr},$$
$$G^i=\frac{1}{4}g^{il}\left(\frac{\partial^2 L}{\partial x^k \partial y^l}y^k-\frac{\partial L}{\partial x^l}+\frac{\partial^2 L}{\partial y^l \partial r}\right), \qquad g_{ij}=\frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}, \tag{1}$$

where L is a Lagrangian function, G^i is called a semispray. The properties of this dynamical system can be described in terms of five KCC geometrical invariants. From the physical point of view, the most interesting invariant is the second one P which is associated with the Jacobi field for geodesics deviation, so it indicates how rapidly different branches of the solution diverge from or converge to the intersection points. Explicitly, the second KCC-invariant can be calculated according to the formula

$$P^i_j=2\frac{\partial G^i}{\partial x^j}+2G^s\frac{\partial N^i_j}{\partial y^s}-\frac{\partial N^i_j}{\partial x^s}y^s-N^i_s N^s_j-\frac{\partial N^i_j}{\partial r}, \; N^i_j=\frac{\partial G^i}{\partial y^j}. \tag{2}$$

In this work we apply the KCC-theory to study systems of differential equations which arise in the theory of electromagnetic and spinor fields on the background of the Schwarzschild spacetime.

Electromagnetic field

In [6] the Maxwell equations were considered on the background of the Schwarzschild spacetime

$$dS^2=\Phi dt^2-\frac{dr^2}{\Phi}-r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{3}$$

where function $\Phi=1-\frac{M}{r}$, and the differential equation system for the radial components was derived after separating the variables in the initial Maxwell equations both in 3-dimensional Majorana-Oppenheimer and 10-demensional Duffin–Kemmer–Petiau approaches. We start with the second order differential equation for the primary radial function F that was obtained in Majorana–Oppenheimer formalism:

$$\frac{d^2F}{dx^2}+\left(\frac{1}{x-1}-\frac{1}{x}\right)\frac{dF}{dx}+\left(M^2\omega^2+\frac{j(j+1)}{x}+\frac{2M^2\omega^2-j(j+1)}{x-1}+\frac{M^2\omega^2}{(x-1)^2}\right)F=0, \tag{4}$$

where $x=r/M$ is a dimensionless coordinate. The spray coefficient G equals

$$G=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x}\right)\frac{dF}{dx}+\frac{1}{2}\left(M^2\omega^2+\frac{j(j+1)}{x}+\frac{2M^2\omega^2-j(j+1)}{x-1}+\frac{M^2\omega^2}{(x-1)^2}\right)F. \tag{5}$$

The second KCC invariant is found in the form

$$P=\frac{-3+4x\left(1-j(j+1)(x-1)+M^2\omega^2x^3\right)}{4(x-1)^2x^2}. \tag{6}$$

In Duffin–Kemmer formalism the equation for another primary radial function f is ([6]):

$$\frac{d^2f}{dr^2}+\left(\frac{\Phi'}{\Phi}+\frac{2}{r}\right)\frac{df}{dr}+\left(\frac{\omega^2}{\Phi^2}-\frac{2\nu^2}{r^2\Phi}+\frac{\Phi'}{\Phi}\frac{1}{r}\right)f=0, \tag{7}$$

where a derivative over r is denoted by a prime. Utilizing the notations $x=r/M$, one finds the second KCC-invariant in the form:

$$P = \frac{-3 + 4x\left(1 - 2\nu^2 (x-1) + M^2\omega^2 x^3\right)}{4(x-1)^2 x^2}, \; 2\nu^2 = j(j+1), \tag{8}$$

which coincides with (6).

Near singular points, the eigenvalue of the second invariant $\Lambda \equiv P$ behaves as follows:

$$x = \frac{r}{M} \to 0 \quad \Lambda \to -\frac{3}{4x^2} < 0; \quad x \to 1 \; (r \to M) \quad \Lambda \to \frac{1 + 4M^2\omega^2}{4(x-1)^2} > 0; \quad x \to \infty \quad \Lambda \to \omega^2 > 0.$$

It indicates that in the vicinity of $x = 0$ (this point is nonphysical) the geodesics converge. Vice versa, near the physical points, Schwarzschild horizon $x = 1$ and at $x \to \infty$, the Jacobi instability exists and the geodesics diverge. The typical behavior of the eigenvalue as a function of the radial coordinate is shown in Fig. 1, *a*.

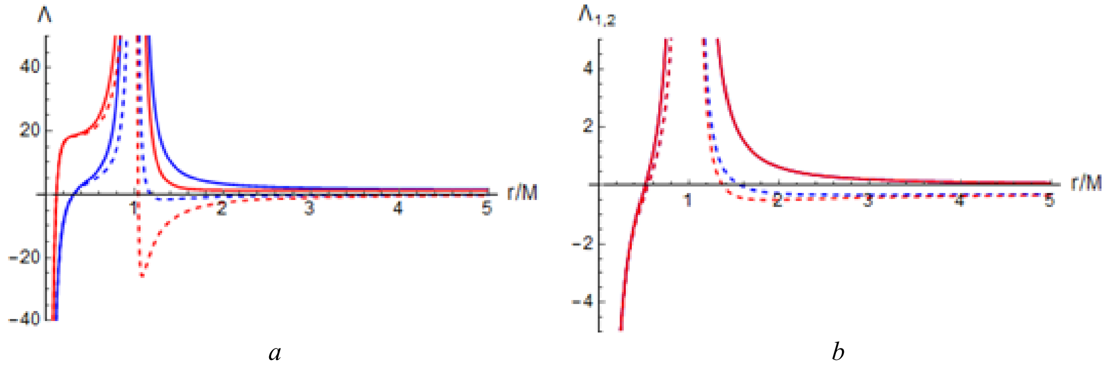

Fig. 1. The behavior of the real parts of the second invariant eigenvalues for the geometrized problem of the electromagnetic (*a*) and spinor (*b*) fields on the background of the Schwarzschild spacetime. The values of parameters used are: *a* – $M = 1$, $\omega = 0.0001$ (dashed curves) and 1.0001 (solid curves), $j = 1$ (blue) and 2 (red); *b* – $m = 1$, $M = 1$, $\varepsilon = 0.0001$ (dashed curves) and 1.0001 (solid curves), $j = 1/2$; blue and red curves correspond to two different eigenvalues Λ_1 and Λ_2

Spinor field

We consider the spin 1/2 particle on the background of the Schwarzschild spacetime. We start from a generally covariant form of the Dirac equation:

$$\left[i\gamma^a \left(e^{\alpha}_{(a)} \frac{\partial}{\partial x^{\alpha}} + \frac{1}{2}\left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\alpha}} \sqrt{-g} e^{\alpha}_{(a)} \right) \right) - m \right] \Psi(x) = 0 \tag{9}$$

in the orthogonal static coordinates $x^{\alpha} = (t, \theta, \varphi, r)$ and tetrad in the Schwarzschild spacetime

$$e^{\beta}_{(a)} = g^{\beta\alpha} e_{(a)\alpha} = \begin{vmatrix} \Phi^{-1/2} & 0 & 0 & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 1/r\sin\theta & 0 \\ 0 & 0 & 0 & \Phi^{1/2} \end{vmatrix}.$$

After separating the variables with diagonalization of the total angular momentum [7, 8] we derive the system of equations

$$\begin{aligned} &\frac{\varepsilon}{\sqrt{1-\frac{M}{r}}} f_3 - i\sqrt{1-\frac{M}{r}} \frac{df_3}{dr} - i\frac{\nu}{r} f_4 - Mf_1 = 0, \quad \frac{\varepsilon}{\sqrt{1-\frac{M}{r}}} f_4 + i\sqrt{1-\frac{M}{r}} \frac{df_4}{dr} + i\frac{\nu}{r} f_3 - Mf_2 = 0, \\ &\frac{\varepsilon}{\sqrt{1-\frac{M}{r}}} f_1 + i\sqrt{1-\frac{M}{r}} \frac{df_1}{dr} - i\frac{\nu}{r} f_2 - Mf_3 = 0, \quad \frac{\varepsilon}{\sqrt{1-\frac{M}{r}}} f_2 - i\sqrt{1-\frac{M}{r}} \frac{df_2}{dr} - i\frac{\nu}{r} f_1 - Mf_4 = 0. \end{aligned} \tag{10}$$

Each complex function $f_i(r)$ we resolved into a sum of real and imaginary parts:

$$f_1 = x_1 + ix_2,\ f_2 = x_3 + ix_4,\ f_3 = x_5 + ix_6,\ f_4 = x_7 + ix_8. \tag{11}$$

By substituting expressions (11) into the system (10) one gets the system of 8 connected differential equation systems of the first order. To bring it to the second order differential system of the type (1) we differentiate each equation over the radial variable. After that the second invariant P^i_j can be directly calculated

$$P^i_j = \begin{pmatrix} \alpha & -\beta & \gamma & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha & 0 & \gamma & 0 & 0 & 0 & 0 \\ \gamma & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & \gamma & -\beta & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & \beta & \gamma & 0 \\ 0 & 0 & 0 & 0 & -\beta & \alpha & 0 & \gamma \\ 0 & 0 & 0 & 0 & \gamma & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & 0 & \gamma & \beta & \alpha \end{pmatrix}, \tag{12}$$

where

$$\alpha = \frac{4r^4(\varepsilon^2 - m^2) + 4Mr\left(m^2r^2 + \nu^2 + 2\right) - 5M^2 - 4\nu^2 r^2}{16r^2(M-r)^2},\ \beta = \frac{\varepsilon M}{4(M-r)^2},\ \gamma = -\frac{\nu}{2r^2\sqrt{1-\frac{M}{r}}}.$$

The eigenvalues Λ_i of the second invariant P^i_j is fourfold degenerated, the different two are determined as

$$\Lambda_{1,2} = \alpha \pm \sqrt{\gamma^2 - \beta^2} = \frac{1}{16}\left(\frac{4m^2r^2 + 4\nu^2 - 2}{r(M-r)} \pm 4\sqrt{-\frac{M^2\varepsilon^2}{(M-r)^4} - \frac{4\nu^2}{r^3(M-r)} + \frac{4r^2\varepsilon^2 + 3}{(M-r)^2} - \frac{5}{r^2}}\right).$$

The dependences of two different eigenvalues $\Lambda_{1,2}$ of the second invariant on the radial variable have been analyzed for different values of energy ε (see Fig. 1, *b*). Near the singular points the eigenvalues behave as follows:

$$r \to 0 \qquad \Lambda_{1,2} \to -\frac{5}{16r^2} < 0,$$

$$r \to M \qquad \Lambda_{1,2} \to \frac{r^2\varepsilon^2 + 3}{4(M-r)^2} \pm \frac{iM\varepsilon}{4(M-r)^2},\quad \mathrm{Re}(\Lambda_{1,2}) \to \frac{r^2\varepsilon^2 + 3}{4(M-r)^2} > 0,$$

$$r \to \infty \qquad \Lambda_{1,2} \to \frac{1}{4}\left(\varepsilon^2 - m^2\right).$$

So, in the vicinity of the Schwarzschild horizon the geodesics diverge at any energy ε, while at $r \to \infty$ the geodesics diverge at $\varepsilon > m$ and converge at $\varepsilon < m$, however the second possibility may be ignored as nonphysical.

The contradictory results have been found out for the problem of the existence of bound states for the Dirac equation in the presence of a black hole [2, 9]. Our results on the KCC-analysis support the conclusion that the fermion bound state on the background of a Schwarzschild black hole are absent as the bound state solution has to be characterized by the convergence of the geodesics flow near the extreme points $r \to M$ and $r \to \infty$. The main argument against the existence of bound states in this system consists of the following: effective potential curves are of a barrier type and do not contain any potential well (see [7]).

Conclusions

The authors applied the KCC-geometrical approach to study the radial systems arising in two problems, electromagnetic and spinor fields on the background of the Schwarzschild spacetime. The stability analysis in terms of the second invariant demonstrates similar behavior of geodesics at $r \to \infty$ for these two systems.

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Authors' contribution

Krylova N.G. performed the study, prepared the manuscript of the article.
Red'kov V.M. stated a research problem, performed the study.

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