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TOTAL PROBABILITY FORMULA FOR VECTOR GAUSSIAN DISTRIBUTIONS

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Abstract. The total probability formula for continuous random variables is the integral of product of two probability density functions that defines the unconditional probability density function from the conditional one. The need for calculation of such integrals arises in many applications, for instant, in statistical decision theory. The statistical decision theory attracts attention due to the ability to formulate the problems in a strict mathematical form. One of the technical problems solved by the statistical decision theory is the problem of dual control that requires calculation of integrals connected with the multivariate probability distributions. The necessary integrals are not available in the literature. One theorem on the total probability formula for vector Gaussian distributions was published by the authors earlier. In this paper we repeat this theorem and prove a new theorem that uses more familiar form of the initial data and has more familiar form of the result. The new form of the theorem allows us to obtain the unconditional mathematical expectation and the unconditional variance-covariance matrix very simply. We also confirm the new theorem by direct calculation for the case of the simple linear regression.

Keywords: total probability formula, vector Gaussian distribution, multivariate integrals, multivariate regression.

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Introduction

The total probability formula for continuous random variables is the integral transformation that transforms the conditional probability density function to the unconditional. The integral transformations of the continuous probability distribution are used in the statistical decision theory [1, 2] and, particularly, in dual control theory [3]. To date, we do not have the required table integrals for multivariate probability distributions [4–6]. The multivariate (vector) normal or Gaussian distribution is of interest that is often used to describe, might approximately, different sets of random variables.

The random vector $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_{k_\Xi})$ with k_Ξ components is distributed according to the normal or Gaussian law if its probability density function has the form

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{k_\Xi} \, |d_\Xi|}} \exp\left(-\frac{1}{2} (\xi - \nu_\Xi)^T d_\Xi^{-1} (\xi - \nu_\Xi) \right), \ \xi \in E^{k_\Xi}, \quad (1)$$

where $\xi^T = (\xi_1, \xi_2, ..., \xi_{k_\Xi})$ is vector-row of the arguments of the probability density function $f(\xi)$, $\nu_\Xi^T = (\nu_{\Xi,1}, \nu_{\Xi,2}, ..., \nu_{\Xi,k_\Xi})$ is the vector-row of the parameters of the probability density function $f(\xi)$, $d_\Xi = (d_{\Xi,i,j})$, $i, j = \overline{1, k_\Xi}$, is the symmetric positive-definite matrix of the parameters of the probability density $f(\xi)$, d_Ξ^{-1} is the matrix inverse to the matrix d_Ξ, $|d_\Xi|$ is the determinant of the matrix d_Ξ, E^{k_Ξ} is the k_Ξ-dimensional Euclidean space, symbol T means transpose. The parameters ν_Ξ and d_Ξ of the distribution (1) are mathematical expectation and variance-covariance matrix of the random vector Ξ respectively [7, 8].

In work [9] there was published the following theorem connected with the vector Gaussian distribution (1).

Theorem 1. (The total probability formula for vector Gaussian distributions). Let $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_{k_\Xi})$ be a row random vector with k_Ξ components, $X^T = (X_1, X_2, ..., X_{k_X})$ be a row random vector with k_X components, $f(\xi)$ be the probability density of the vector Ξ, $f(x/\xi)$ be the condition probability density of the vector X, E^{k_Ξ} be the k_Ξ-dimensional Euclidean space. If in the total probability formula

$$f(x) = \int_{E^{k_\Xi}} f(x/\xi) f(\xi) d\xi, \tag{2}$$

the probability density $f(x/\xi)$ is represented in the form

$$f(x/\xi) = \frac{1}{\sqrt{(2\pi)^{k_X} |d_X|}} \exp\left(-\frac{1}{2}\xi^T S \xi + V^T \xi - \frac{1}{2} W \right) \tag{3}$$

and the probability density $f(\xi)$ is represented in the form

$$f(\xi) = \frac{1}{\sqrt{(2\pi)^{k_\Xi} |d_\Xi|}} \exp\left(-\frac{1}{2}\xi^T d_\Xi^{-1} \xi + \nu_\Xi^T d_\Xi^{-1} \xi - \frac{1}{2} \nu_\Xi^T d_\Xi^{-1} \nu_\Xi \right), \tag{4}$$

then integral (2) (the total probability formula) is defined by the following expression:

$$f(x) = \int_{E^{k_\Xi}} f(x/\xi) f(\xi) d\xi = \frac{1}{\sqrt{(2\pi)^{k_X} |d_\Xi| |A| |d_X|}} \exp\left(\frac{1}{2} B^T A^{-1} B - \frac{1}{2} C \right), \tag{5}$$

where

$$A = d_\Xi^{-1} + S, \tag{6}$$

$$B = d_\Xi^{-1} \nu_\Xi + V, \tag{7}$$

$$C = \nu_\Xi^T d_\Xi^{-1} \nu_\Xi + W. \tag{8}$$

In addition to the theorem 1, we will prove the following theorem 2 which can be more useful in some application compared with the theorem 1.

The new theorem on the total probability formula for vector Gaussian distributions

Theorem 2. Let $\Xi^T = (\Xi_1, \Xi_2, ..., \Xi_{k_\Xi})$ be a random row vector with k_Ξ components, $X^T = (X_1, X_2, ..., X_{k_X})$ be a random row vector with k_X components, $f(\xi)$ be the probability density function of the vector Ξ, $f(x/\xi)$ be the condition probability density function of the vector X, E^{k_Ξ} be the k_Ξ-dimensional Euclidean space. If in the total probability formula

$$f(x)=\int_{E^{k_\Xi}} f(x/\xi)f(\xi)d\xi, \qquad (9)$$

the probability density function $f(x/\xi)$ has the form

$$f(x/\xi)=\frac{1}{\sqrt{(2\pi)^{k_X}\,|d_X|}}\exp\left(-\frac{1}{2}(x-h\xi)^T d_X^{-1}(x-h\xi)\right), \qquad (10)$$

where $h=(h_{i,j})$, $i=\overline{1,k_X}$, $j=\overline{1,k_\Xi}$, is the $(k_X\times k_\Xi)$-matrix, and the probability density function $f(\xi)$ has the form

$$f(\xi)=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|d_\Xi|}}\exp\left(-\frac{1}{2}(\xi^T-\nu_\Xi^T)d_\Xi^{-1}(\xi-\nu_\Xi)\right),$$

then integral (9) (the total probability formula) is defined by the following expression:

$$f(x)=\int_{E^{k_\Xi}} f(x/\xi)f(\xi)d\xi=\frac{1}{\sqrt{(2\pi)^{k_X}\,|D_X|}}\exp\left(-\frac{1}{2}(x-h\nu_\Xi)^T D_X^{-1}(x-h\nu_\Xi)\right), \qquad (11)$$

where $D_X=d_X+hd_\Xi h^T$.

Proof. We will use the theorem 1 and represent the functions $f(x/\xi)$ and $f(\xi)$ in the form (3), (4) respectively:

$$f(\xi)=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|d_\Xi|}}\exp\left(-\frac{1}{2}(\xi^T-\nu_\Xi^T)d_\Xi^{-1}(\xi-\nu_\Xi)\right)=$$

$$=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|d_\Xi|}}\exp\left(-\frac{1}{2}\xi^T d_\Xi^{-1}\xi+\nu_\Xi^T d_\Xi^{-1}\xi-\frac{1}{2}\nu_\Xi^T d_\Xi^{-1}\nu_\Xi\right),$$

$$f(x/\xi)=\frac{1}{\sqrt{(2\pi)^{k_X}\,|d_X|}}\exp\left(-\frac{1}{2}(x-h\xi)^T d_X^{-1}(x-h\xi)\right)=$$

$$=\frac{1}{\sqrt{(2\pi)^{k_X}\,|d_X|}}\exp\left(-\frac{1}{2}x^T d_X^{-1}x+x^T d_X^{-1}h\xi-\frac{1}{2}\xi^T h^T d_X^{-1}h\xi\right).$$

In accordance with the theorem 1 we have the following notations:
$S=h^T d_X^{-1}h$, $V^T=x^T d_X^{-1}h$, $W=x^T d_X^{-1}x$.
Then in the formulas (5)–(8)

$$A=d_\Xi^{-1}+S=d_\Xi^{-1}+h^T d_X^{-1}h,$$

$$B=d_\Xi^{-1}\nu_\Xi+V=d_\Xi^{-1}\nu_\Xi+h^T d_X^{-1}x,\quad B^T=\nu_\Xi^T d_\Xi^{-1}+x^T d_X^{-1}h,$$

$$C=\nu_\Xi^T d_\Xi^{-1}\nu_\Xi+W=\nu_\Xi^T d_\Xi^{-1}\nu_\Xi+x^T d_X^{-1}x,$$

$$B^T A^{-1}B=(\nu_\Xi^T d_\Xi^{-1}+x^T d_X^{-1}h)A^{-1}(d_\Xi^{-1}\nu_\Xi+h^T d_X^{-1}x)=$$

$$=\nu_\Xi^T d_\Xi^{-1}A^{-1}d_\Xi^{-1}\nu_\Xi+2x^T d_X^{-1}hA^{-1}d_\Xi^{-1}\nu_\Xi+x^T d_X^{-1}hA^{-1}h^T d_X^{-1}x=$$

$$=\nu_\Xi^T d_\Xi^{-1}A^{-1}d_\Xi^{-1}\nu_\Xi+2\nu_\Xi^T d_\Xi^{-1}A^{-1}h^T d_X^{-1}x+x^T d_X^{-1}hA^{-1}h^T d_X^{-1}x.$$

Let us substitute the last expressions into the total probability formula (5) and perform some transformations. We will get:

$$f(x)=\int_{E^{k_\Xi}} f(x/\xi)f(\xi)d\xi=M_X\exp\left(\frac{1}{2}B^T A^{-1}B-\frac{1}{2}C\right)=$$

$$=M_X\exp\left(\frac{1}{2}(\nu_\Xi^T d_\Xi^{-1}A^{-1}d_\Xi^{-1}\nu_\Xi+2x^T d_X^{-1}hA^{-1}d_\Xi^{-1}\nu_\Xi+x^T d_X^{-1}hA^{-1}h^T d_X^{-1}x)-\frac{1}{2}(\nu_\Xi^T d_\Xi^{-1}\nu_\Xi+x^T d_X^{-1}x)\right)=$$

$$= M_X \exp\left(\frac{1}{2}x^T d_X^{-1} h A^{-1} h^T d_X^{-1} x + x^T d_X^{-1} h A^{-1} d_\Xi^{-1} \nu_\Xi + \frac{1}{2}\nu_\Xi^T d_\Xi^{-1} A^{-1} d_\Xi^{-1} \nu_\Xi - \frac{1}{2}\nu_\Xi^T d_\Xi^{-1} \nu_\Xi - \frac{1}{2} x^T d_X^{-1} x\right) =$$

$$= M_X \exp\left(-\frac{1}{2}x^T (d_X^{-1} - d_X^{-1} h A^{-1} h^T d_X^{-1}) x + \nu_\Xi^T (d_\Xi^{-1} A^{-1} h^T d_X^{-1}) x - \frac{1}{2}\nu_\Xi^T (d_\Xi^{-1} - d_\Xi^{-1} A^{-1} d_\Xi^{-1}) \nu_\Xi\right), \quad (12)$$

where $M_X = \dfrac{1}{\sqrt{(2\pi)^{k_X} \, |d_\Xi| \, |A| \, |d_X|}}$.

Let us now reduce the expression (12) to the form (11). We will use for this the known identity [10]

$$(A \pm BCB^T)^{-1} = A^{-1} - A^{-1}B(B^T A^{-1} B \pm C^{-1})^{-1} B^T A^{-1}. \quad (13)$$

According this identity we have

$$A^{-1} = (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} = d_\Xi - d_\Xi h^T (d_X + h d_\Xi h^T)^{-1} h d_\Xi. \quad (14)$$

Substituting the right side of the expression (14) into the first summand in (12) instead of $A^{-1} = (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1}$, we obtain the multiplier $(d_X^{-1} - d_X^{-1} h (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} h^T d_X^{-1})$ which, in accordance with the identity (13), is $D_X^{-1} = (d_X + h d_\Xi h^T)^{-1}$.

Substituting the right side of the expression (14) into the third summand in (12) instead of $A^{-1} = (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1}$, we obtain the multiplier which is equal to $h^T D_X^{-1} h$:

$$(d_\Xi^{-1} - d_\Xi^{-1} A^{-1} d_\Xi^{-1}) = d_\Xi^{-1} - d_\Xi^{-1}(d_\Xi - d_\Xi h^T (d_X + h d_\Xi h^T)^{-1} h d_\Xi) d_\Xi^{-1} =$$
$$= d_\Xi^{-1} - (d_\Xi^{-1} d_\Xi d_\Xi^{-1} - d_\Xi^{-1} d_\Xi h^T (d_X + h d_\Xi h^T)^{-1} h d_\Xi d_\Xi^{-1}) =$$
$$= d_\Xi^{-1} - (d_\Xi^{-1} - h^T (d_X + h d_\Xi h^T)^{-1} h) = d_\Xi^{-1} - d_\Xi^{-1} + h^T D_X^{-1} h = h^T D_X^{-1} h.$$

Finally, we have to proof the following identity for the second summand in (12):

$$h^T D_X^{-1} = d_\Xi^{-1} A^{-1} h^T d_X^{-1},$$

or

$$h^T (d_X + h d_\Xi h^T)^{-1} = d_\Xi^{-1} (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} h^T d_X^{-1}.$$

Applying the formula (13) to the left side of the last equality, we get the following equality:

$$h^T d_X^{-1} - h^T d_X^{-1} h (d_\Xi^{-1} + h^T d_X H)^{-1} h^T d_X^{-1} = d_\Xi^{-1} (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} h^T d_X^{-1}.$$

Multiplying this equality by the d_X on the right and transferring the right side to the left we get:

$$h^T - h^T d_X^{-1} h (d_\Xi^{-1} + h^T d_X H)^{-1} h^T - d_\Xi^{-1} (d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} h^T =$$
$$= h^T - (h^T d_X^{-1} h + d_\Xi^{-1})(d_\Xi^{-1} + h^T d_X^{-1} h)^{-1} h^T = h^T - h^T = 0.$$

Therefore, we have the following expression

$$f(x) = \int_{E^{k_\Xi}} f(x/\xi) f(\xi) d\xi = M_X \exp\left(-\frac{1}{2} x^T D_X^{-1} x + \nu_\Xi^T h^T D_X^{-1} x - \frac{1}{2} \nu_\Xi^T h^T D_X^{-1} h \nu_\Xi\right),$$

which is the expression (11). This completes the proof of the theorem 2.

It should be noted that the expression $h\xi$ in (10) is the conditional mathematical expectation of the random vector X (the regression function of X on Ξ). The result of the theorem 2 in the form of the expression (11) shows that the unconditional mathematical expectation of the random vector X is equal to $\nu_X = h\nu_\Xi$ and the unconditional variance-covariance matrix of the random vector X is equal to $D_X = d_X + h d_\Xi h^T$.

The total probability formula for scalar Gaussian distributions

Let ξ, x and h in the theorem 2 are scalars. This means that we consider a simple linear Gaussian regression of x on ξ in the form of $h\xi$. We will obtain for this case unconditional probability density function $f(x)$ by the direct calculation using the theorem 1. Then, in accordance with the theorem 1, we have

$$A=\frac{1}{d_\Xi}+\frac{h^2}{d_X}=\frac{d_X+h^2 d_\Xi}{d_\Xi d_X},\ A^{-1}=\frac{d_\Xi d_X}{d_X+h^2 d_\Xi},$$

$$f(x)=M_x\exp\left(-\frac{1}{2}x^T(d_X^{-1}-d_X^{-2}A^{-1}h^2)x+\nu_\Xi^T(d_\Xi^{-1}A^{-1}hd_X^{-1})x-\frac{1}{2}\nu_\Xi^T(d_\Xi^{-1}-d_\Xi^{-2}A^{-1})\nu_\Xi\right).$$

Substituting of the expression for A^{-1} into the expression for the $f(x)$ we get the multiplier in the first summand of the $f(x)$

$$(d_X^{-1}-d_X^{-2}A^{-1}h^2)=\frac{1}{d_X}-\frac{h^2 d_\Xi}{d_X(d_X+h^2 d_\Xi)}=\frac{(d_X+h^2 d_\Xi)-h^2 d_\Xi}{d_X(d_X+h^2 d_\Xi)}=\frac{1}{d_X+h^2 d_\Xi}=D_X^{-1}$$

and the multiplier in the second summand of the $f(x)$

$$(d_\Xi^{-1}A^{-1}hd_X^{-1})=\frac{h}{d_\Xi}\frac{d_\Xi d_X}{(d_X+h^2 d_\Xi)}\frac{1}{d_X}=\frac{h}{(d_X+h^2 d_\Xi)}=hD_X^{-1}$$

and the multiplier in the third summand of the $f(x)$

$$(d_\Xi^{-1}-d_\Xi^{-1}A^{-1}d_\Xi^{-1})=(\frac{1}{d_\Xi}-\frac{1}{d_\Xi}\frac{d_\Xi d_X}{(d_X+h^2 d_\Xi)}\frac{1}{d_\Xi})=\frac{1}{d_\Xi}-\frac{d_X}{(d_X+h^2 d_\Xi)d_\Xi}=$$

$$=\frac{d_X+h^2 d_\Xi-d_X}{(d_X+h^2 d_\Xi)d_\Xi}=\frac{h^2 d_\Xi}{(d_X+h^2 d_\Xi)d_\Xi}=\frac{h^2}{(d_X+h^2 d_\Xi)}=h^2 D_X^{-1}$$

and normalization constant

$$M_X=\frac{1}{\sqrt{(2\pi)^n|d_\Xi||A||d_X|}}=\frac{1}{\sqrt{2\pi(d_X+h^2 d_\Xi)}}=\frac{1}{\sqrt{2\pi D_X}}.$$

Therefore

$$f(x)=\frac{1}{\sqrt{2\pi D_X}}\exp\left(-\frac{x^2}{2D_X}+\frac{h\nu_\Xi x}{D_X}-\frac{h^2\nu_\Xi^2}{2D_X}\right)=\frac{1}{\sqrt{2\pi D_X}}\exp\left(-\frac{1}{2D_X}(x-h\nu_\Xi)^2\right).$$

We can see that the scalar case satisfies the theorem 2.

A simple example

We will consider some stochastic controlled object which is described by the conditional probability density function $f(x/\xi,u)$, where x is the output scalar variable of the object, u is the input vector variable of the object and ξ is the vector of the parameters of the object. As a rule, it is the Gaussian (normal) probability density function:

$$f(x/\xi,u)\sim N(\varphi(\xi,u),d_X), \quad (15)$$

where $\varphi(\xi,u)$ is the regression function of x on u, ξ is the vector of the parameters of the regression function, d_X is the variance of the internal noise of the object. The description (15) can be represented in the form

$$X=\varphi(\xi,u)+\mathrm{E}, \quad (16)$$

where E is the random variable with Gaussian distribution $N(0,d_\mathrm{E})$ and $d_\mathrm{E}=d_X$ is the variance of the random variable E. An object with description (15) or (16) is called a multiple regression object. The class of the functions represented in the form

$$\varphi(\xi,u)=\sum_{j=1}^{m}h_j(u)\xi_j, \quad (17)$$

where $h_j(u)$, $j = 1,2,...,m$ are some functions called basis functions, that are usually used for description of the multiple regression function. The function (17) can be written in a vector form as a dot product of the vectors h and ξ:

$$\varphi(\xi, u) = h^T \xi = \xi h^T , \quad (18)$$

where $h^T = h^T(u) = (h_1(u), h_2(u),..., h_m(u))$ is the row vector of the basis funtions and $\xi^T = (\xi_1, \xi_2,..., \xi_m)$ is the row vector of the parameters. Let us consider the regression function of two variables in the following form: $\varphi = \alpha + \beta u_1 + \gamma u_2 + \tau u_1^2$.
Then we will have this regression function in the form of (18) with

$$h^T = (1, u_1, u_2, u_1^2), \ \xi^T = (\alpha, \beta, \gamma, \tau). \quad (19)$$

Now we suppose that the vector of the parameters $\xi^T = (\alpha, \beta, \gamma, \tau)$ is random vector with Gaussian probability density function $f(\xi) \sim N(\nu_\Xi, d_\Xi)$ having mean value ν_Ξ and variance-covariance matrix d_Ξ. The unconditional moments $\nu_{X,k}$ of the output variable of the controlled object are of interest:

$$\nu_{X,k} = \int_{E^{k_\Xi}} x^k f(x/\xi) f(\xi) d\xi . \quad (20)$$

They can be easily defined by the theorem 2 as the moments of the Gaussian distribution $N(h^T \nu_\Xi, d_X + h^T d_\Xi h)$ [7]. For instance, in accordance with the theorem 2 we get:
$\nu_{X,1} = h^T \nu_\Xi$, $\nu_{X,2} = (h^T \nu_\Xi)^2 + d_X + h^T d_\Xi h$, where h is defined as (19) and d_Ξ is the (4×4) variance-covariance matrix of the random vector ξ (19).

Conclusion

The results obtained in theorems 1 and 2 are aimed at solving the dual control problems formulated in works [11, 12]. The sequence of the control actions in dual control of the multivariate stochastic objects is defined by the functional equations of the dynamic programming which contain the integral like (20) subjected to calculation [11]. One of the practical examples is the task of the optimal allowance distribution as the task of the dual control considered in the work [12].

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Authors' contribution

Mukha V.S. performed the work on the scientific supervision, set the statement of the problem and prepared the article for publication.

Kako N.F. performed the work on the scientific content of the article.

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